

PARAMETRIZATION OF  $\epsilon$ -RATIONAL CURVES: ERROR ANALYSIS

SONIA L. RUEDA AND JUANA SENDRA

ABSTRACT. In [5] the authors present an algorithm to parametrize approximately  $\epsilon$ -rational curves, and they show in 2 examples that the Hausdorff distance, w.r.t. to the Euclidean distance, between the input and output curves is small. In this paper, we analyze this distance for a whole family of curves randomly generated and we automatize the strategy used in [5]. We find a reasonable upper bound of the Hausdorff distance between each input and output curve of the family.

## INTRODUCTION

The problem of the approximate parametrization of algebraic plane curves goes as follows: given a plane algebraic curve  $\mathcal{C}$  (that is the perturbation of a rational plane curve) and a tolerance  $\epsilon > 0$ , we want to find a new curve  $\overline{\mathcal{C}}$ , being rational, as well as a rational parametrization of it such that  $\mathcal{C}$  and  $\overline{\mathcal{C}}$  are at certain small distance dependent on  $\epsilon$ .

In [2] and [3] it was seen how to approximately parametrize algebraic plane curves and algebraic surfaces having an  $\epsilon$ -singularity of maximum multiplicity. In [5], using the techniques of  $\epsilon$ -points developed in [2] and [4], we extended the results in [2] to the general case of algebraic affine plane curves. More precisely, we provide in [5] the approximate parametrization algorithm which given  $\mathcal{C}$  returns an approximate parametrization  $\overline{\mathcal{P}}(t)$  of the curve  $\overline{\mathcal{C}}$ .

A natural question arises, the closeness between the input and output curves of the algorithm. In our case, this closeness notion is given by the Hausdorff distance. That is, we say that the input and output curves are close if their Hausdorff distance (as real curves) is small related to the tolerance. We proved in [5] that the Hausdorff distance between  $\mathcal{C}$  and  $\overline{\mathcal{C}}$  is finite.

In this paper, apply the approximate parametrization algorithm to a family of curves  $\mathcal{F}$  of degree 4 which was randomly generated. For each curve  $\mathcal{C}$  in  $\mathcal{F}$  we compute a bound  $\mathcal{B}$  of the Hausdorff distance between  $\mathcal{C}$  and  $\overline{\mathcal{C}}$ . For all of the curves we get  $\mathcal{B} \leq 2.2$  and we obtain evidences that the actual distance is experimentally  $\leq 0.1$ .

The paper is organized as follows. We recall the approximate parametrization algorithm in Section 1. In Section 2 we explain how the family  $\mathcal{F}$  of curves of degree 4 was generated. The last section is devoted to the analysis of the distance between the the curves of  $\mathcal{F}$  and their approximate parametrizations output by our algorithm.

The following terminology will be used throughout the paper.  $\|\cdot\|$  and  $\|\cdot\|_2$  denote the polynomial  $\infty$ -norm and the usual unitary norm in  $\mathbb{C}^2$ , respectively.  $|\cdot|$  denotes the module in  $\mathbb{C}$ . The partial derivatives of a polynomial  $g \in \mathbb{C}[x, y]$  are denoted by  $g^{\vec{v}} := \partial^{i+j}g/\partial^i x \partial^j y$ ,

where  $\vec{v} = (i, j) \in \mathbb{N}^2$ ; we assume that  $g^{\vec{v}} = g$ . Moreover, for  $\vec{v} = (i, j) \in \mathbb{N}^2$ ,  $|\vec{v}| = i + j$ . Also,  $\vec{e}_1 = (1, 0)$ , and  $\vec{e}_2 = (0, 1)$ .

## 1. RECALLING THE PARAMETRIZATION ALGORITHM

In this section we recall the algorithm presented in [5] as well as its main properties; see [5] for further details. We start with a fixed tolerance  $\epsilon$ , and with the implicit equation  $f(x, y)$  of a real plane algebraic curve  $\mathcal{C}$  of exact degree  $d$ , which is the perturbation of a rational curve.  $\mathcal{C}$  is supposed to satisfy that:

- (1) the degree  $d$  of  $f$  is proper; this means that there exists a partial derivative of  $f$ , of order  $d$ , that in module is strictly bigger than  $\epsilon\|f\|$ .
- (2)  $f$  is  $\epsilon$ -irreducible.
- (3)  $\mathcal{C}$  has  $d$  different points at infinity, and it does not pass through  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ .

Conditions (1) and (2) guarantee that, under the tolerance, we really have an irreducible curve of degree  $d$ . Condition (3) ensures that  $\mathcal{C}$  is either compact (as a subset of  $\mathbb{R}^2$ ) or it follows real asymptotes. Therefore, we are excluding curves having a parabolic behavior. The requirement on  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  is technical and it can be achieved by performing a suitable and orthogonal linear change of coordinates.

The theoretical argumentation of the algorithm is as follows. First, the notion of exact singularity is replaced by the concept of  $\epsilon$ -singularity, similarly with the notions of exact multiplicity and  $\epsilon$ -multiplicity. Here, the first complication appears since the number of  $\epsilon$ -singularities is bigger than (expected) in the exact case; probably due to the perturbation. In order to deal with this difficulty we associate to each  $\epsilon$ -singularity a radius, and hence we see it as an Euclidean disk. Next, we introduce an equivalence relation on the set of disks and we define the  $\epsilon$ -singular clusters as the equivalence classes. Then, we define the  $\epsilon$ -multiplicity of the cluster as the maximum of the  $\epsilon$ -multiplicities within the class, and we take as canonical representant of the cluster an  $\epsilon$ -singularity where the  $\epsilon$ -multiplicity of the cluster is achieved. In this situation, we say that  $\mathcal{C}$  is  $\epsilon$ -rational if the clusters satisfy the well-known genus formula of the exact case. More precisely, if  $\{\mathcal{C}luster_{r_i}(Q_i)\}_{i=1,\dots,s}$  is the cluster decomposition ( $Q_i$  denotes the canonical representant and  $r_i$  the  $\epsilon$ -multiplicity of the cluster), we say that  $\mathcal{C}$  is  $\epsilon$ -(affine) rational if

$$(d-1)(d-2) - \sum_{i=1}^s r_i(r_i-1) = 0.$$

Now, let us assume that  $\mathcal{C}$  is  $\epsilon$ -rational, and let us see how the approximate parametrization algorithm proceeds. The basic idea is, as in the exact case (see [7]), to construct a suitable linear system of curves of degree  $d-2$ . More precisely, if

$$\{\mathcal{C}luster_{r_i}(Q_i)\}_{i=1,\dots,s}, \quad Q_i = (q_{i1} : q_{i2} : 1)$$

is the cluster decomposition, we compute  $d-3$  simple  $\epsilon$ -points on  $\mathcal{C}$ , say  $\{P_1, \dots, P_{d-3}\}$  with  $P_i = (p_{i1} : p_{i2} : 1)$ . Again, we associate to each  $\epsilon$ -point an Euclidean disk via a radius, and we apply the equivalence relation. If, somehow, any  $P_i$  is identified with another  $P_j$  or with a singular cluster, we replace  $P_i$  by a new  $\epsilon$ -point. In this situation, we consider the

effective divisor

$$D = \sum_{i=1}^s r_i Q_i + \sum_{i=1}^{d-3} P_i$$

and the (exact) linear system  $\mathcal{H}$  of curves of degree  $(d-2)$  given by  $D$ . That is,  $\mathcal{H}$  is the linear system of curves of degree  $d-2$  having  $Q_i$  as  $(r_i-1)$ -base points, and  $P_i$  as simple base points. If we were working exactly, all intersection points in  $\mathcal{H} \cap \mathcal{C}$  would be fixed (namely those points in  $D$ ) with the exception of one point that would provide the parametrization. Indeed, in the exact case, the parametrization would be

$$\left( \frac{\text{Res}_y(H(x, y, 1), f)}{\prod_{i=1}^s (x - q_{i,1})^{r_i(r_i-1)} \prod_{i=1}^{d-3} (x - p_{i,1})}, \frac{\text{Res}_x(H(x, y, 1), f)}{\prod_{i=1}^s (y - q_{i,2})^{r_i(r_i-1)} \prod_{i=1}^{d-3} (y - p_{i,2})} \right),$$

being  $H(x, y, z)$  the homogeneous polynomial defining  $\mathcal{H}$ . In the approximate case, instead of the exact division above, we take the quotient of the Euclidean division of each numerator by the corresponding denominator.

The output curve derived from this process has the same structure at infinity as the input curve and the same degree, see [5], Theorem 4.5. These properties will play a fundamental role in the error analysis (see Section 3). We outline the algorithm derived from the above ideas.

- (1) Compute the singular cluster decomposition  $\{\mathcal{C}luster_{r_i}(Q_i)\}_{i=1,\dots,s}$ ; say  $Q_i = (q_{i,1} : q_{i,2} : 1)$ .
- (2) If  $\sum_{i=1}^s r_i(r_i-1) \neq (d-1)(d-2)$ , RETURN " $\mathcal{C}$  is not (affine)  $\epsilon$ -rational". If  $s = 1$  one may apply the algorithm in [2].
- (3) Compute  $(d-3)$   $\epsilon$ -simple points  $\{P_j\}_{1 \leq j \leq d-3}$  of  $\mathcal{C}$ . Take the points over  $\mathbb{R}$ , or as conjugate complex points. After each point computation check that it is not in the cluster of the others (including the clusters of  $Q_i$ ); if this fails take a new one. Say  $P_i = (p_{i,1} : p_{i,2} : 1)$ .
- (4) Determine the linear system  $\mathcal{H}$  of degree  $(d-2)$  given by the divisor  $\sum_{i=1}^s r_i Q_i + \sum_{i=1}^{d-3} P_i$ . Let  $H(t, x, y, z) = H_1(x, y, z) + tH_2(x, y, z)$  be its defining polynomial.
- (5) If  $[\gcd(F(x, y, 0), H_1(x, y, 0)) \neq 1]$  and  $[\gcd(F(x, y, 0), H_2(x, y, 0)) \neq 1]$  replace  $H_2$  by  $H_2 + \rho_1 x^{d-2} + \rho_2 y^{d-2}$ , where  $\rho_1, \rho_2$  are real and strictly smaller than  $\epsilon$ . Say that  $\gcd(F(x, y, 0), H_2(x, y, 0)) = 1$ ; similarly in the other case.
- (6)  $S_1(x, t) = \text{Res}_y(H(x, y, 1), f)$  and  $S_2(y, t) = \text{Res}_x(H(x, y, 1), f)$ .
- (8)  $A_1(x) = \prod_{i=1}^s (x - q_{i,1})^{r_i(r_i-1)} \prod_{i=1}^{d-3} (x - p_{i,1})$ ,  
 $A_2(y) = \prod_{i=1}^s (y - q_{i,2})^{r_i(r_i-1)} \prod_{i=1}^{d-3} (y - p_{i,2})$ .
- (9) For  $i = 1, 2$  compute the quotient  $B_i$  of  $S_i$  by  $A_i$  w.r.t. either  $x$  or  $y$ .
- (10) If the content of  $B_1$  w.r.t  $x$  or the content of  $B_2$  w.r.t.  $y$  does depend on  $t$ , RETURN "degenerate case".
- (11) Determine the root  $\bar{p}_1(t)$  of  $B_1$ , as a polynomial in  $x$ , and the root  $\bar{p}_2(t)$  of  $B_2$ , as a polynomial in  $y$ .
- (12) RETURN  $\bar{\mathcal{P}}(t) = (\bar{p}_1(t), \bar{p}_2(t))$ .

## 2. GENERATING A FAMILY OF $\epsilon$ -RATIONAL CURVES

In this section, we generate the family of curves that will be used in the error analysis. We fix three points  $P_1 = (2 : 0 : 1)$ ,  $P_2 = (0 : 0 : 1)$  and  $P_3 = (1 : 1 : 1)$  in  $\mathbb{P}^2(\mathbb{C})$  and we consider the linear system of curves of degree 4 defined by the divisor  $2P_1 + 2P_2 + 2P_3$ . Its defining polynomial is

$$G(x, y, z, u_1, \dots, u_6) = u_2 y^2 z^2 + u_3 y^3 z + u_4 y^4 + u_5 x y z^2 - (2u_2 + 3u_3 + 4u_4 + \frac{1}{2}u_5 + 2u_6) x y^2 z + u_6 x y^3 + u_1 x^2 z^2 + (-\frac{3}{2}u_5 + 2u_3 + 4u_4 + 2u_6 - u_1) x^2 y z + (u_2 + u_3 + \frac{1}{2}u_5 + \frac{1}{4}u_1 + u_4) x^2 y^2 - u_1 x^3 z + (\frac{1}{2}u_5 - u_3 - 2u_4 - u_6 + \frac{1}{2}u_1) x^3 y + \frac{1}{4}u_1 x^4.$$

Note that for every specialization of  $u_i$  such that  $G(x, y, z, u_1, \dots, u_6)$  is irreducible, we get an (exact) rational curve.

Now, for  $j = 1, \dots, 6$  and  $i = 1, \dots, 10$  let  $r_{ij}$  be a random integer number in the interval  $[0, 100]$ . We obtain 60 different polynomials  $G_{ij}(x, y, z)$ ,  $j = 1, \dots, 6$ ,  $i = 1, \dots, 10$  setting

$$u_k = \begin{cases} (\frac{r_{ij}}{100})^i & \text{if } k = j \\ 1 & \text{if } k \neq j \end{cases} \quad k = 1, \dots, 6$$

in  $G(x, y, z, u_1, \dots, u_6)$ . Given  $i \in \{1, \dots, 6\}$  and  $j \in \{1, \dots, 10\}$  we obtain a random perturbation  $g_{ij}(x, y) \in \mathbb{R}[x, y]$  of  $G_{ij}(x, y, 1)$  as follows

$$g_{ij}(x, y) = G_{ij}(x, y, 1) + \epsilon \frac{r_1}{100}(x + y) + \epsilon^2 \frac{r_2}{100}(x^2 + xy + y^2) + \epsilon^3 \frac{r_3}{100}(x^3 + x^2y + xy^2 + y^3)$$

where  $r_1, r_2, r_3$  are integer numbers taken randomly in the interval  $[0, 100]$  and  $\epsilon = \frac{1}{100}$ . The polynomials  $g_{ij}(x, y)$ ,  $j = 1, \dots, 6$ ,  $i = 1, \dots, 10$  have proper degree 4 and define 60 curves  $\mathcal{C}_{ij}$  verifying  $(1 : 0 : 0), (0 : 1 : 0) \notin \mathcal{C}_{ij}^h$  ( $\mathcal{C}_{ij}^h$  is the projective closure of  $\mathcal{C}_{ij}$ ) and such that they have 4 different points at infinity. Therefore, each of 60 curves satisfies the hypothesis required in parametrization algorithm.

Using the parametrization algorithm described in Section 1, we conclude that 28 of the 60 curves are  $\epsilon$ -rational. We show those curves in Fig. 1. An statistical error analysis was given in [6]. The precise equations of  $\mathcal{C}_{ij}$  as well as the parametrizations provided by the algorithm can be found in

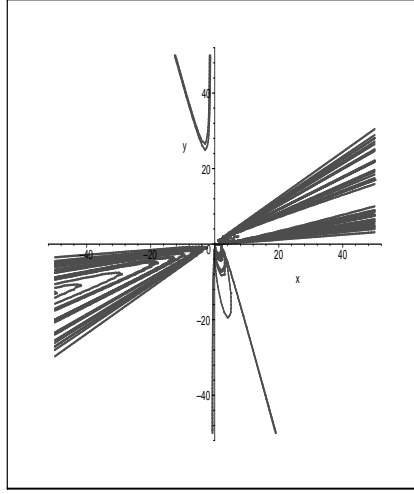
<http://www.aq.upm.es/Departamentos/Matematicas/srueda/fam4.pdf>.

## 3. THEORETICAL STRATEGY FOR THE ERROR ANALYSIS

We describe the theoretical strategy for the error analysis that will be employed in the last section. Let  $\epsilon > 0$  be the tolerance,  $\mathcal{C}$  the input curve and  $\overline{\mathcal{C}}$  the output curve provided by the approximate parametrization algorithm. Also, let  $f(x, y)$  and  $\overline{f}(x, y)$  be the defining polynomials of  $\mathcal{C}$  and  $\overline{\mathcal{C}}$ , respectively. Moreover, let  $\overline{\mathcal{P}}(t)$  be the parametrization of  $\overline{\mathcal{C}}$  output by the algorithm. Let  $\mathcal{C}^{\mathbb{R}}$  and  $\overline{\mathcal{C}}^{\mathbb{R}}$  denote the real part of  $\mathcal{C}$  and  $\overline{\mathcal{C}}$ , respectively.

Now, we consider the Hausdorff distance  $H$ , induced by Euclidean distance  $d$  in the real plane. That is, for  $A, B \subset \mathbb{R}^2 \setminus \{\emptyset\}$

$$H(A, B) = \max\{\sup_{a \in A}\{d(a, B)\}, \sup_{b \in B}\{d(b, A)\}\}$$

FIGURE 1. Plot of the 28  $\epsilon$ -rational curves  $\mathcal{C}_{ij}$  randomly generated.

where  $d(a, B) = \inf_{b \in B} \{d(a, b)\}$ , and by convection  $H(\emptyset, \emptyset) = 0$  and, for  $\emptyset \neq A \subset X$ ,  $H(A, \emptyset) = \infty$  (see [1], for further details).

The most relevant fact, for our analysis, is that  $H(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}}) < \infty$  (see [5], Lemma 6.1) because we want to bound  $H(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}})$ . For this purpose, we will proceed in a similar manner to Section 6 in [5], not only for a couple of examples, but for the whole family of  $\epsilon$ -rational curves randomly generated in Section 2. More precisely, we consider the normal line to  $\overline{\mathcal{C}}$  at the generic point  $\overline{\mathcal{P}}(t)$ , say  $\mathcal{L}_1(t, s)$ , as well as the normal line to  $\mathcal{C}$  at the generic point  $(a, b) \in \mathcal{C}^{\mathbb{R}}$ , say  $\mathcal{L}_2(a, b, s)$ . Moreover, we introduce the polynomials

$$\mathcal{D}_1(t, s) = f(\mathcal{L}_1(t, s)) \in \overline{\mathbb{R}(t)}[s], \quad \mathcal{D}_2(a, b, s) = \overline{f}(\mathcal{L}_2(a, b, s)) \in \mathbb{C}(\overline{\mathcal{C}})[s],$$

where  $\overline{\mathbb{R}(t)}$  denotes the algebraic closure of  $\mathbb{R}(t)$  and  $\mathbb{C}(\overline{\mathcal{C}})$  the field of rational functions over  $\overline{\mathcal{C}}$ . In addition, for every  $t_0 \in \mathbb{R}$ , such that  $\mathcal{D}_1(t_0, s)$  is well defined and has real roots, and for every  $(a_0, b_0) \in \mathcal{C}^{\mathbb{R}}$ , such that  $\mathcal{D}_2(a_0, b_0, s)$  is well defined and has real roots, we take

$$\begin{aligned} \rho_1^{\mathbb{R}}(t_0) &= \min\{|s_0| / \mathcal{D}_1(t_0, s_0) = 0 \text{ and } s_0 \in \mathbb{R}\}, \\ \rho_2^{\mathbb{R}}(a_0, b_0) &= \min\{|s_0| / \mathcal{D}_2(a_0, b_0, s_0) = 0 \text{ and } s_0 \in \mathbb{R}\}. \end{aligned}$$

Then, the supremum of  $\rho_1^{\mathbb{R}}(t), \rho_2^{\mathbb{R}}(a, b)$  provides an upper bound of the Hausdorff distance; at least for those subsets of both curves where the considered minimums are well defined.

Because of computational difficulties, in our analysis, instead of computing  $\rho_1^{\mathbb{R}}(t), \rho_2^{\mathbb{R}}(a, b)$ , we will study

$$\begin{aligned} \rho_1(t_0) &:= \min\{|s_0| / \mathcal{D}_1(t_0, s_0) = 0\}, \\ \rho_2(a_0, b_0) &:= \min\{|s_0| / \mathcal{D}_2(a_0, b_0, s_0) = 0\}. \end{aligned}$$

These quantities bound  $d(\overline{\mathcal{P}}(t_0), \mathcal{C})$  and  $d((a_0, b_0), \overline{\mathcal{C}})$  respectively (here, we understand that  $d$  is the unitary distance in  $\mathbb{C}^2$ ), instead of  $d(\overline{\mathcal{P}}(t_0), \mathcal{C}^{\mathbb{R}})$  and  $d((a_0, b_0), \overline{\mathcal{C}}^{\mathbb{R}})$ . So, for those subsets of both curves, where the corresponding polynomials are well defined, we bound

$$\Delta(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}}) := \max\{\sup_{t \in \mathbb{R}} \{d(\overline{\mathcal{P}}(t), \mathcal{C})\}, \sup_{(a, b) \in \overline{\mathcal{C}}^{\mathbb{R}}} \{d((a, b), \mathcal{C})\}\}.$$

For this purpose, in the next section, we will follow the next steps:

- (1) We compute a bound of  $\sup_{t \in \mathbb{R}} \{d(\overline{\mathcal{P}}(t), \mathcal{C})\}$ , by applying Corollary 6.2 in [5] that ensures that

$$d(\overline{\mathcal{P}}(t), \mathcal{C}) \leq \min \left\{ \binom{n}{i} \left| \frac{A_0(t)}{A_i(t)} \right|^{\frac{1}{i}} \text{ where } A_i(t) \neq 0 \text{ and } 1 \leq i \leq n \right\},$$

where  $\mathcal{D}_1(t, s) = A_n(t)s^n + \dots + A_0(t)$ .

- (2) In order to bound  $\sup_{(a,b) \in \overline{\mathcal{C}}^{\mathbb{R}}} \{d((a,b), \mathcal{C})\}$  one may apply the same corollary as in (1). However, it implies to maximize a bivariate rational function under the constrain  $f(x, y) = 0$ . This can be done, for instance, using Lagrange multipliers. Nevertheless, in practice, this is unfeasible. Instead, we estimate the bound by taking a lattice of points  $(a, b) \in \overline{\mathcal{C}}$  where we bound  $d((a, b), \overline{\mathcal{C}})$ .
- (3) Note that the quantity  $\Delta(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}})$  gives information on how close every real point on each of the curves is of a complex point on the other curve. However,  $\Delta(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}}) \leq H(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}})$ . To avoid this difficulty, in our analysis, we will look for empirical evidences indicating that the computed bound of  $\Delta(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}})$  also bounds  $H(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}})$ ; for that we test empirically that, in our computations,  $\rho_1(t_0) = \rho_1^{\mathbb{R}}(t_0)$ ,  $\rho_2(a_0, b_0) = \rho_2^{\mathbb{R}}(a_0, b_0)$ .
- (4) It may happen for some  $t_0 \in \mathbb{R}$  that  $\rho_1^{\mathbb{R}}(t_0)$  is not well defined or simply that the bound it provides of  $d(\overline{\mathcal{P}}(t_0), \mathcal{C}^{\mathbb{R}})$  is not satisfactory. Observe that to bound  $d(\overline{\mathcal{P}}(t_0), \mathcal{C}^{\mathbb{R}})$  we can use the intersection of any line through  $\overline{\mathcal{P}}(t_0)$  with  $\mathcal{C}$ . So in some cases we will also proceed in the following way.

Let us consider the line at the generic point  $\overline{\mathcal{P}}(t)$  in the direction given by  $h \in \mathbb{R}$

$$\mathcal{L}_h(t, s) = \left( \overline{p}_1(t) + s \frac{2h}{h^2 + 1}, \overline{p}_2(t) + s \frac{h^2 - 1}{h^2 + 1} \right).$$

We introduce the polynomial  $\mathcal{D}_h(t, s) = f(\mathcal{L}_h(t, s)) \in \overline{\mathbb{R}(t, h)}[s]$ . For a fixed  $h_0 \in \mathbb{R}$  and for every  $t_0 \in \mathbb{R}$ , such that  $\mathcal{D}_{h_0}(t_0, s)$  is well defined and has real roots,  $d(\overline{\mathcal{P}}(t_0), \mathcal{C}^{\mathbb{R}}) \leq \rho_{h_0}^{\mathbb{R}}(t_0)$ , where

$$\rho_{h_0}^{\mathbb{R}}(t_0) = \min\{|s_0| / \mathcal{D}_{h_0}(t_0, s_0) = 0 \text{ and } s_0 \in \mathbb{R}\}.$$

Thus, the supremum of  $\rho_{h_0}^{\mathbb{R}}(t)$  and  $\rho_2^{\mathbb{R}}(a, b)$  provides an upper bound of the Hausdorff distance. Then for  $t_0 \in \mathbb{R}$  such that  $\mathcal{D}_{h_0}(t_0, s)$  is well defined, we may study

$$\rho_{h_0}(t_0) := \min\{|s_0| / \mathcal{D}_{h_0}(t_0, s_0) = 0\}$$

which is an upper bound of  $d(\overline{\mathcal{P}}(t_0), \mathcal{C})$  and plays the role of  $\rho_1(t_0)$  in the previous steps.

#### 4. EXECUTION OF THE ERROR ANALYSIS

Let  $\mathcal{F}$  be the family of 28  $\epsilon$ -rational curves of degree 4 defined in Section 3. For each curve  $\mathcal{C}$  in  $\mathcal{F}$  we explain next how the bound of  $\Delta(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}})$  was computed and show evidences that this bound is also an upper bound of the Hausdorff distance  $H(\mathcal{C}^{\mathbb{R}}, \overline{\mathcal{C}}^{\mathbb{R}})$ .

4.1. **Bound of  $\sup_{t \in \mathbb{R}} \{d(\overline{\mathcal{P}}(t), \mathcal{C})\}$ .** Let us denote by  $\Lambda$  the domain of  $\mathcal{D}_1(t_0, s)$ . If the curve is compact then the polynomial  $\mathcal{D}_1(t_0, s)$  is well defined for every  $t_0 \in \mathbb{R}$ , otherwise it is not defined for two real poles  $\beta_1, \beta_2$  of  $\overline{\mathcal{P}}(t)$ . In the family  $\mathcal{F}$  only two curves are compact.

In order to bound  $\sup_{t \in \mathbb{R}} \{d(\overline{\mathcal{P}}(t), \mathcal{C})\}$ , we obtain an upper bound of  $\rho_1(t)$  when  $t \in \Lambda$ . For this purpose we maximize the functions  $R_1(t) = 4 \left| \frac{A_0(t)}{A_1(t)} \right|$  and  $R_2(t) = \binom{4}{2} \left| \frac{A_0(t)}{A_2(t)} \right|^{\frac{1}{2}}$  (see step (1) in Section 3) in  $\Lambda$  as follows.

Let  $\alpha_1$  and  $\alpha_2$  be the real roots of the denominator of  $R_1(t)$ . For all the curves in  $\mathcal{F}$ ,  $\alpha_1$  and  $\alpha_2$  are not real roots of the denominator of  $R_2(t)$ . Let  $I_i$ ,  $i = 1, 2$  be an interval isolating  $\alpha_i$  from  $\alpha_j$ ,  $j \neq i$  and from the real poles of  $R_2(t)$ . Observe that  $R_1(t)$  and  $R_2(t)$  are continuous in  $\mathbb{R} \setminus (I_1 \cup I_2)$  and in the adherence  $\overline{I_1 \cup I_2}$  of  $I_1 \cup I_2$ , respectively. We compute

$$\mathcal{B}_1 = \max\{R_1(t) \mid t \in \mathbb{R} \setminus (I_1 \cup I_2)\} \quad \text{and} \quad \mathcal{B}_2 = \max\{R_2(t) \mid t \in \overline{I_1 \cup I_2}\}.$$

Then  $\mathcal{B} = \max\{\mathcal{B}_1, \mathcal{B}_2\}$  is an upper bound of  $\rho_1(t)$  in  $\Lambda$ .

The last column of the next table contains the computed bound  $\mathcal{B}$  for each one of the curves  $\mathcal{C}_i$ ,  $i = 1, \dots, 28$  of the family  $\mathcal{F}$ .

Table 1:

$i$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}$
1	0.3012751472	1.784885546	1.784885546
2	0.1680336313	0.8228821157	0.8228821157
3	0.2209183305	1.143210796	1.143210796
4	0.2457462218	1.388890611	1.388890611
5	0.4775061243	1.471164469	1.471164469
6	0.1854050321	0.9172323537	0.9172323537
7	0.3392516285	1.238494405	1.238494405
8	0.1687631697	0.9278483955	0.9278483955
9	0.4481254299	1.345341665	1.345341665
10	0.1706747632	1.252669418	1.252669418
11	0.4336254993	1.328637472	1.328637472
12	0.04502452088	0.7028506083	0.7028506083
13	0.2511290220	1.849173820	1.849173820
14	0.7973544750	0.5426224779	0.7973544750
15	1.947190823	1.201605769	1.947190823
16	0.1658993167	2.124343900	2.124343900
17	0.06346428265	1.634020447	1.634020447
18	1.401107905	1.830395156	1.830395156
19	1.690902532	0.9731554792	1.690902532
20	1.590285558	1.659392056	1.659392056
21	0.1543267485	0.9761129297	0.9761129297
22	0.8247063503	0.8459442935	0.8459442935
23	0.2452763324	0.6140170288	0.6140170288
24	0.08434729326	0.7159251709	0.7159251709
25	0.6464253153	2.150679036	2.150679036



Table 1: continued

26	3.604620794	1.418101314	3.604620794
27	0.08779082555	0.7809344831	0.7809344831
28	0.3673519642	1.773877016	1.773877016

We will improve next the bound given for curves  $\mathcal{C}_{16}$  and  $\mathcal{C}_{26}$ . For a fixed  $h_0 \in \mathbb{R}$  we can write  $\mathcal{D}_{h_0}(t, s) = B_n(t)s^n + \dots + B_0(t)$  to which Corollary 6.3 in [5] applies. Hence we can obtain an upper bound  $\mathcal{B}^{h_0}$  of  $\sup_{t \in \mathbb{R}} \{d(\overline{\mathcal{P}}(t), \mathcal{C})\}$  maximizing the new functions

$$4 \left| \frac{B_0(t)}{B_1(t)} \right| \quad \text{and} \quad \binom{4}{2} \left| \frac{B_0(t)}{B_2(t)} \right|^{\frac{1}{2}}$$

in the domain of  $\mathcal{D}_{h_0}(t, s)$ . As described earlier for  $R_1(t)$  and  $R_2(t)$  we obtain respectively  $\mathcal{B}_1^{h_0}$  and  $\mathcal{B}_2^{h_0}$ .

For curves  $\mathcal{C}_{16}$  and  $\mathcal{C}_{26}$  we computed the upper bound  $\mathcal{B}^{h_0}$  of  $\sup_{t \in \mathbb{R}} \{d(\overline{\mathcal{P}}(t), \mathcal{C})\}$  using different values of  $h_0$  and we found bounds improving the ones given earlier for the values of  $h_0$  shown in the next table.

Table 2:

$i$	$h_0$	$\mathcal{B}_1^{h_0}$	$\mathcal{B}_2^{h_0}$	$\mathcal{B}^{h_0}$
16	-1	0.5870746534	1.287063889	1.287063889
26	$\frac{1}{20}$	0.2525792337	0.001139009266	1.178706930

**4.2. Empirical bound of  $\sup_{(a,b) \in \overline{\mathcal{C}}^{\mathbb{R}}} \{d((a,b), \mathcal{C})\}$ .** In this section we estimate the bound of  $\sup_{(a,b) \in \overline{\mathcal{C}}^{\mathbb{R}}} \{d((a,b), \mathcal{C})\}$ . We estimate the bound by taking a lattice of points  $(a, b) \in \overline{\mathcal{C}}$  where we bound  $d((a, b), \overline{\mathcal{C}})$  estimating  $\rho_2^{\mathbb{R}}(a, b)$ . We show evidences for  $\rho_2^{\mathbb{R}}(a, b)$  being small and for  $\rho_2(a, b) = \rho_2^{\mathbb{R}}(a, b)$ .

If the curve is not compact, first we analyze the behavior of the input and output curves through the real asymptotes. Let us suppose that  $\mathcal{C}$  is a non compact curve in  $\mathcal{F}$  and let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be its real asymptotes. By [5], Corollary 4.6 the real asymptotes of  $\mathcal{C}$  and  $\overline{\mathcal{C}}$  are parallel lines so the Hausdorff distance between them can be easily computed. Let  $\overline{\mathcal{L}}_1$  and  $\overline{\mathcal{L}}_2$  be the real asymptotes of  $\overline{\mathcal{C}}$  parallel to  $\mathcal{L}_1$  and  $\mathcal{L}_2$  respectively. We the value of

$$\eta = \max\{H(\mathcal{L}_1, \overline{\mathcal{L}}_1), H(\mathcal{L}_2, \overline{\mathcal{L}}_2)\}$$

for all the non compact curves of  $\mathcal{F}$  in the next table. Then we proceed as follows:

- (1) For each negative integer  $i$  we compute the set  $\Omega_i$  of intersections of  $\mathcal{C}^{\mathbb{R}}$  with the line  $x = i$ . We obtain  $m_i^{\mathbb{R}} := \max\{\rho_2^{\mathbb{R}}(a, b) \mid (a, b) \in \Omega_i\}$  and  $m_i := \max\{\rho_2(a, b) \mid (a, b) \in \Omega_i\}$ , and we check that  $m_i^{\mathbb{R}} = m_i$ .
- (2) We repeat the previous step until

$$\min\{|\rho_2^{\mathbb{R}}(i, b) - H(\mathcal{L}_1, \overline{\mathcal{L}}_1)|, |\rho_2^{\mathbb{R}}(i, b) - H(\mathcal{L}_2, \overline{\mathcal{L}}_2)| \mid (i, b) \in \Omega_i\} < \varepsilon.$$

- (3) Let  $\tau_1$  be the smallest value of  $i$  until termination of this process.

We perform this experiment also for each positive integer  $i$  to obtain in this case the highest value  $\tau_2$  such that the inequality in step (2) holds. At the same time we check that



$m_i^{\mathbb{R}} = m_i$  with positive integers  $i = 1, \dots, \tau_2$ . The same process is repeated for  $y = j$ , to obtain the negative and positive integers  $\tau_3, \tau_4$ , respectively such that

$$\min\{|\rho_2^{\mathbb{R}}(a, j) - H(\mathcal{L}_1, \overline{\mathcal{L}}_1)|, |\rho_2^{\mathbb{R}}(a, j) - H(\mathcal{L}_2, \overline{\mathcal{L}}_2)| \mid (a, j) \in \Omega^j\} < \varepsilon$$

where  $\Omega^j$  is the set of intersections of  $\mathcal{C}^{\mathbb{R}}$  with the line  $y = j$ . Let  $(m^j)^{\mathbb{R}} := \max\{\rho_2^{\mathbb{R}}(a, b) \mid (a, b) \in \Omega^j\}$  and  $m^j := \max\{\rho_2(a, b) \mid (a, b) \in \Omega^j\}$ . We also check that  $(m^j)^{\mathbb{R}} = m^j$  with integers  $j = \tau_3, \dots, -1, 1, \dots, \tau_4$ . Let  $m := \max\{m_i, m^j \mid i = \tau_1, \dots, -1, 1, \dots, \tau_2, j = \tau_3, \dots, -1, 1, \dots, \tau_4\}$ .

Let  $[\tau_1, \tau_2] \times [\tau_3, \tau_4]$ . We empirically consider that out of the compact  $[\tau_1, \tau_2] \times [\tau_3, \tau_4]$ , the curves behave as the asymptotes, and the empirical bound of  $\sup_{(a,b) \in \overline{\mathcal{C}}^{\mathbb{R}}} \{d((a, b), \mathcal{C})\}$  in  $[\tau_1, \tau_2] \times [\tau_3, \tau_4]$  is  $m$ .

The next table shows the compact set  $[\tau_1, \tau_2] \times [\tau_3, \tau_4]$  obtained for  $\varepsilon = 10^{-6}$  in all the curves except for numbers 25 and 28 for which we took  $\varepsilon = 10^{-5}$ . The reason being that we run out of memory before reaching the box outside of which the curves behaved like the asymptotes with  $\varepsilon = 10^{-5}$ .

Table 3:

$i$	$[\tau_1, \tau_2]$	$[\tau_3, \tau_4]$	$m$	$\eta$
1	[-3434, 3428]	[-2069, 2066]	0.04474051996	0.002685992105
2	[-6730, 6732]	[-3833, 3835]	0.01909150476	0.007250422655
3	[-2120, 2120]	[-1030, 1031]	0.02523781400	0.0008098244306
4	[-2485, 2492]	[-1370, 1374]	0.03061351675	0.0006391265474
5	[-4157, 4160]	[-2000, 2003]	0.03559861599	0.0007910063013
6	[-11115, 11089]	[-4663, 4642]	0.02312094507	0.005224665954
7	[-22154, 22121]	[-9370, 9334]	0.03942395261	0.01044227346
8	[-11207, 11214]	[-4756, 4761]	0.02118572337	0.005399010161
9	[-28777, 28684]	[-11887, 11825]	0.05222449790	0.01308589690
10	[-1922, 1925]	[-1063, 1065]	0.02633886946	0.0003845822760
11	[-4555, 4552]	[-1912, 1912]	0.03226327617	0.001373344655
12	[-5017, 4997]	[-2114, 2107]	0.01246202377	0.002415851115
14	[-797, 780]	[-242, 235]	0.01907109331	0.001277331144
15	[-4809, 4774]	[-1410, 1396]	0.07529126612	0.005987329671
16	[-3841, 3866]	[-15344, 15340]	0.05528214429	0.01084165622
17	[-139, 16]	[-955, 832]	0.03702449872	0.008212462120
19	[-13558, 13560]	[-6428, 6446]	0.03068560995	0.01035036823
20	[-4290, 4291]	[-2287, 2288]	0.04051580912	0.001538264208
21	[-14465, 14520]	[-5275, 5300]	0.01835471004	0.005065273865
22	[-2286, 2282]	[-982, 982]	0.01911087028	0.0002824836230
23	[-946, 940]	[-400, 399]	0.01154559037	0.0004413541720
24	[-157, 3724]	[-1215, 1213]	0.01530235590	0.0001629862393
25	[-18910, 18839]	[-5882, 5863]	0.08880952924	0.03464418857
26	[-3400, 3398]	[1769, 1769]	0.03324362713	0.001139009266
27	[-308, 5274]	[1659, 1642]	0.01648328102	0.0002351747177
28	[-6279, 6210]	[-2001, 1982]	0.03876376237	0.01250853150

If the curve  $\mathcal{C}$  is compact we consider a compact set  $[\tau_1, \tau_2] \times [\tau_3, \tau_4]$  containing  $\mathcal{C}^{\mathbb{R}}$ . Then we compute  $m$  as previously described checking also that  $m_i^{\mathbb{R}} = m_i, (m^j)^{\mathbb{R}} = m^j$  with  $i = \tau_1, \dots, -1, 1, \dots, \tau_2, j = \tau_3, \dots, -1, 1, \dots, \tau_4$ .

Table 4:

$i$	$[\tau_1, \tau_2]$	$[\tau_3, \tau_4]$	$m$
13	$[-9/512, 8041/1024]$	$[-20057/1024, 2117/1024]$	0.04595703645
18	$[-5/512, 592745/512]$	$[-8723/1024, 304847/1024]$	0.09228397972

**4.3. Empirical evidences.** Now, we perform some empirical tests to show evidences that  $\rho_1^{\mathbb{R}}(t)$  is smaller than the upper bound  $\mathcal{B}$  of  $\sup_{t \in \mathbb{R}} \{d(\overline{\mathcal{P}}(t), \mathcal{C})\}$  given in Section 4.1. First, let  $D_1(s) = \lim_{t \rightarrow \pm\infty} \mathcal{D}_1(t, s)$ . Then, for every curve of the family  $\mathcal{F}$  let

$$\chi = \min\{|s_0| / D_1(s_0) = 0 \text{ and } s_0 \in \mathbb{R}\}.$$

We checked that  $\chi = \min\{|s_0| / D_1(s_0) = 0\}$  in all cases. Since the roots of a polynomial depend continuously on its coefficients, for every  $\delta > 0$  there exists  $K > 0$  such that for all  $|t_0| > K$  there is a root  $s_0$  of  $\mathcal{D}_1(t_0, s)$  with  $\|\chi - s_0\|_2 < \delta$ . It may happen that these roots are all complex. However, in our example, we see that  $\rho_1((-10)^k) = \rho_1^{\mathbb{R}}((-10)^k)$  for  $k = 1, \dots, 20$ . Let  $\chi_1 = \min\{\rho_1^{\mathbb{R}}((-10)^k) \mid k = 1, \dots, 20\}$  and  $\chi_2 = \max\{\rho_1^{\mathbb{R}}((-10)^k) \mid k = 1, \dots, 20\}$ . We show these computations in the next table.

Table 5:

$i$	$\chi$	$\chi_1$	$\chi_2$
1	0.001918863706	0.001918568088	0.001922644324
2	0.004169957700	0.004161970065	0.004170583666
3	0.0006994105148	0.0006993543405	0.0007001275847
4	0.0006662568567	0.0006659169716	0.0006706114665
5	0.001185494963	0.001184862374	0.001191850748
6	0.0002278441391	0.0002266218453	0.0002434582060
7	0.004927221514	0.004922425041	0.004975309877
8	0.001581018062	0.001580991493	0.001581020714
9	0.003893937744	0.003759724534	0.008421643120
10	0.0004038643651	0.0004036487366	0.0004066244124
11	0.001324473283	0.001324364325	0.001325864354
12	0.0006513574356	0.0006500882228	0.0006514569249
13	0.004018853976	0.004018487265	0.004023539023
14	0.001399552125	0.001399552100	0.001399552395
15	0.006540790926	0.006540789722	0.006540806320
16	0.01873734902	0.001829670103	0.03745550824
17	0.0003018645266	0.0002989553843	0.0003020916294
18	0.003558391244	0.003557440039	0.003570581899

Table 5: continued

$i$	$\chi$	$\chi_1$	$\chi_2$
19	0.0005206627618	0.0005145010748	0.0005996180488
20	0.001782196531	0.001779734571	0.001782429636
21	0.001157562963	0.001155811228	0.001179946145
22	0.001096140685	0.001095297129	0.001096220492
23	0.0004670320359	0.0004670115956	0.0004670339681
24	0.001057066460	0.001056738383	0.001061264741
25	0.06749573507	0.005148188392	0.06814019159
26	0.001564435599	0.001563775730	0.001572903801
27	0.001268468809	0.001267989563	0.001274605947
28	0.0005550274941	0.0004502065315	0.0005632086626

Observe that for most of the curves of the family  $\mathcal{F}$  the bound  $\mathcal{B}$  computed in Section 5.1 was reached at  $\mathcal{B}_2$ . Let us suppose that the isolating intervals of the real poles of  $R_1(t)$  are  $I_1 = [t_{11}, t_{12}]$  and  $I_2 = [t_{21}, t_{22}]$ . Then if  $\Gamma_2 = \{t \in \mathbb{R} / R'_2(t) = 0\}$  then

$$\mathcal{B}_2 = \max\{R_2(t) / t \in \Gamma_2 \cup \{t_{11}, t_{12}, t_{21}, t_{22}\}\}.$$

Furthermore, only for curve  $\mathcal{C}_{25}$  the set  $\Gamma_2$  is nonempty. We show in the next table that  $\nu = \max\{\rho_1^{\mathbb{R}}(t_{11}), \rho_1^{\mathbb{R}}(t_{12}), \rho_1^{\mathbb{R}}(t_{21}), \rho_1^{\mathbb{R}}(t_{22})\}$  is very small compared to the value of  $\mu = \max\{R_2(t_{11}), R_2(t_{12}), R_2(t_{21}), R_2(t_{22})\}$  and also we compare them with  $\mathcal{B}_2$ .

Table 6:

$i$	$\mathcal{B}_2$	$\mu$	$\nu$
1	1.784885546	1.784885546	0.01895037003
2	0.8228821157	0.8228821157	0.003187256654
3	1.143210796	1.143210796	0.007932988669
4	1.388890611	1.388890611	0.01141905470
5	1.471164469	1.471164469	0.01307915412
6	0.9172323537	0.9172323537	0.004427051081
7	1.238494405	1.238494405	0.008019188716
8	0.9278483955	0.9278483955	0.004590148599
9	1.345341665	1.345341665	0.009661551521
10	1.252669418	1.252669418	0.009307873805
11	1.328637472	1.328637472	0.01086359027
12	0.7028506083	0.7028506083	0.002603616572
13	1.849173820	1.849173820	0.01407639891
14	0.5426224779	0.5426224779	0.002434019507
15	1.201605769	1.201605769	0.01147127202
16	2.124343900	2.124343900	0.01663838395
17	1.634020447	1.634020447	0.01149183389
18	1.830395156	1.830395156	0.01586113294
19	0.9731554792	0.9731554792	0.006071538946
20	1.659392056	1.659392056	0.01644172109
21	0.9761129297	0.9761129297	0.005852977427
22	0.8459442935	0.8459442935	0.004712176306
23	0.6140170288	0.6140170288	0.002448022081

Table 6: continued

$i$	$\mathcal{B}_2$	$\mu$	$\nu$
24	0.7159251709	0.7159251709	0.006089518478
25	2.150679036	2.150679036	0.05229161121
26	1.418101314	1.418101314	0.01218396938
27	0.7809344831	0.7809344831	0.007553005576
28	1.773877016	1.773877016	0.01971360528

For curves  $\mathcal{C}_i$ ,  $i = 14, 15, 19, 26$  the bound  $\mathcal{B}$  is achieved in  $\mathcal{B}_1$ . Let  $\Gamma_1 = \{t \in \mathbb{R} / R'_1(t) = 0\}$ , then

$$\mathcal{B}_1 = \max\{\{R_1(t) / t \in \Gamma_1 \cup \{t_{11}, t_{12}, t_{21}, t_{22}\}\} \cup \{\lim_{t \rightarrow \pm\infty} R_1(t)\}\}.$$

In fact, in those curves  $\mathcal{B}_1$  is equal to  $\max\{R_1(t) / t \in \Gamma_1\}$ .

For each real pole of  $R_1(t)$  as well as for each real critical value of  $R_1(t)$ , we consider a sequence of isolating intervals  $J_k$  of length  $1/10^{k+5}$ , we take the middle point  $t_k$ , and we analyze  $\rho_1(t_k), \rho_1^{\mathbb{R}}(t_k)$ . After a certain  $k_1$  the sequences become stable, let  $E_{k_1}$  be the set containing  $\rho_1^{\mathbb{R}}(t_{k_1})$  for the  $k_1$ th element of each one of the sequences constructed for each real pole of  $R_1(t)$ . Let  $\gamma_1 = \max E_{k_1}$ . Similarly, after a certain  $k_2$  the sequences for the real critical values of  $R_1(t)$  become stable. Let  $E_{k_2}$  be the set containing  $\rho_1^{\mathbb{R}}(t_{k_2})$  for the  $k_2$ th element of each one of the sequences constructed for each real critical values of  $R_2(t)$ . We call  $\gamma_2 = \max E_{k_2}$ . If the curve is not compact, we also perform this experiment for each of the two real poles  $\beta_1, \beta_2$  of  $\overline{\mathcal{P}}(t)$ . The sequences to  $\beta_1, \beta_2$  become stable after a certain  $k_3$  and we get a set  $E_{k_3}$  and  $\gamma_3 = \max E_{k_3}$ .

Table 7:

i	$\gamma_1$	$\gamma_2$	$\gamma_3$
1	0.05760637790	0.07103885930	0.002685991017
2	0.03680694646	0.04030133537	0.007250422354
3	0.03985346560	0.05219935882	0.0008098246652
4	0.04358984318	0.06256427563	0.0006391277723
5	0.05222536651	$\max\{0.009579373594, s_{c_1}\}$	0.0007910068620
6	0.03997031881	0.03468777768	0.005224667630
7	0.05420521510	0.06230995426	0.01044227515
8	0.03842889832	0.03215152707	0.005399009187
9	0.06425437603	0.08158624352	0.01308589929
10	0.03857067352	0.03800183984	0.0003845830022
11	0.05050891041	$\max\{0.007217929868, s_{c_5}\}$	0.001373344987
12	0.02590356741	0.01124398891	0.002415851766
13	0.07620545140	0.05039734679	
14	0.02750848667	$\max\{0.01846235849, s_{c_5}\}$	0.001277332295
15	0.06016762755	0.2971598318 (*)	0.005987330302
16	0.09714515451	0.03765565867	0.01084165864

Table 7: continued

17	0.06901149454	0.01524196447	0.008212460400
18	0.07011950234	$\max\{0.01367185130, s_{c_5}\}$	
19	0.05269958334	0.2771889722 (*)	0.01035036672
20	0.05744093868	$\max\{0.01461432871, s_{c_2}\}$	0.001538264277
21	0.05357323512	0.03486848807	0.005065270849
22	0.04568079074	$\max\{0.009770701627, s_{c_2}\}$	0.0002824834209
23	0.02825492360	0.06513903382	0.0004413527932
24	0.05459990275	0.01907914217	0.0001629875241
25	0.09628252969	0.1341804080	0.03464419053
26	0.05149574516	1.431046152 (*)	0.001139009002
27	0.06133149851	0.01990495615	0.0002351734340
28	0.09919600166	0.09135201774	0.01250853076

4.4. **Behaviour of  $\rho_1^{\mathbb{R}}(t)$ .** In the grate majority of our computations  $\rho_1^{\mathbb{R}}(t)$  is defined and  $\rho_1^{\mathbb{R}}(t) = \rho_1(t)$ . In some cases for a given  $t_0 \in \mathbb{R}$  the set  $\{|s_0| / \mathcal{D}_1(t_0, s_0) = 0 \text{ and } s_0 \in \mathbb{R}\}$  is empty or  $\rho_1^{\mathbb{R}}(t)$  and  $\rho_1(t)$  happen to be different. Then it should be taken into consideration that there exists  $h_0 \in \mathbb{R}$  such that  $\rho_{h_0}^{\mathbb{R}}(t_0) < \infty$ .

Let  $s_c$  denote  $\{\rho_1^{\mathbb{R}}(t_k)\}$  for the sequence  $\{t_k\}$  to the critical point  $c$  of  $R_1(t)$ . For curves number 5, 11, 18, 20 and 22 the sequence  $s_c$  could not be computed for some critical point  $c$  of  $R_1(t)$ . For example, curve  $\mathcal{C}_5$  has 6 critical points and the sequence  $s_{c_1}$  for critical point  $c_1$  was not defined. We write  $\max\{0.009579373594, s_{c_1}\}$  where 0.009579373594 is the maximum of the values at which the rest of the sequences stabilized.

The results for  $\gamma_2$  marked with (\*) indicate that  $\rho_1^{\mathbb{R}}(t_k) \neq \rho_1(t)$  for the sequence  $\{t_k\}$  to one of the critical points of  $R_1(t)$ . For curve number 26 there are 6 critical values of  $R_1(t)$  and  $1.357539211 \leq |\rho_1^{\mathbb{R}}(t_k) - \rho_1(t_k)| \leq 1.357539223$  for the sequence  $\{t_k\}$  to the critical value  $c_5$ .

For each one of the curves highlighted and for the critical point of  $R_1(t)$  where  $\rho_1^{\mathbb{R}}(t)$  did not behave properly (there was only one of those points of each curve) we proceed as follows. We consider a sequence of isolating intervals  $J_k$  of length  $1/10^{k+5}$ , we take the middle point  $t_k$ , and we analyze  $\rho_{h_0}^{\mathbb{R}}(t_k)$  for different values of  $h_0$ . The next table shows the values of  $h_0$  giving good results together with the value  $\rho_{h_0}^{\mathbb{R}}(t_{k'_2})$  at which the sequence became stable and the new maximum  $\gamma'_2$ .

Table 8:

$i$	$h_0$	$\rho_{h_0}^{\mathbb{R}}(t_{k'_2})$	$\gamma'_2$
5	1	0.05275311956	0.05275311956
11	1	0.05120027918	0.05120027918
14	1	0.06065146651	0.06065146651
15	1	0.09855121223	0.09855121223
18	1	0.08937284288	0.08937284288
19	4/5	0.1099404739	0.1099404739
20	1	0.07070538339	0.07070538339
22	1	0.06827542251	0.06827542251
26	1	0.07787356026	0.07787356026

**Acknowledgements.** The authors deeply thank Sonia Pérez-Díaz and J. Rafael Sendra for many useful discussions on the topics treated in this paper.

## REFERENCES

- [1] Aliprantis C.D., Border K.C. (2006). *Infinite Dimensional Analysis*. Springer Verlag.
- [2] Pérez-Díaz, S., Sendra, J., Sendra, J.R., (2004). *Parametrizations by Lines of Approximate Algebraic Curves*. Theoretical Computer Science on Algebraic-Numeric Algorithms, 315/2-3, pp. 627-650.
- [3] Pérez-Díaz, S., Sendra, J., Sendra, J.R., (2005). *Parametrizations by Lines of Approximate Algebraic Surfaces*. Computer Aided Geometric Design, 22/2, pp. 147-181.
- [4] Pérez-Díaz, S., Sendra, J., Sendra, J.R., (2006). *Distance Bounds of  $\epsilon$ -Points on Hypersurfaces*. Theoretical Computer Science, 359/1-3. pp. 344 - 368.
- [5] Pérez-Díaz, S., Rueda S.L., Sendra, J., Sendra, J.R., (2009). *Approximate Parametrization of Plane Algebraic Curves by Linear Systems of Curves*. Computer Aided Geometric Design, (to appear).
- [6] Pérez-Díaz, S., Rueda S.L., Sendra, J., Sendra, J.R., (2009). *Parametrization of Epsilon-Rational Curves*. ACM. Proceedings of SCN'09 Kyoto, Japan pp 199-200.
- [7] Sendra J.R., Winkler J.R., Pérez-Díaz S. (2007). *Rational Algebraic Curves: A Computer Algebra Approach*. Springer-Verlag Heidelberg, in series Algorithms and Computation in Mathematics. Vol. 22.

Dpto. de Matemática Aplicada, E.T.S. Arquitectura, Univ. Politécnica de Madrid, Spain  
*E-mail address:* sonialuisa.rueda@upm.es

Dpto. de Matemática Aplicada a la I.T. de Telecomunicación, E.U.I.T.Telecomunicación, Univ. Politécnica de Madrid, Spain  
*E-mail address:* jsendra@euitt.upm.es